

SOME SHARP INEQUALITIES FOR THE TOADER-QI MEAN

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*Dedicated to my father Xin-Jiang Yang.*ABSTRACT. The Toader-Qi mean of positive numbers a and b defined by

$$TQ(a, b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta$$

is related to the modified Bessel function of the first kind. In this paper, we present several properties of this mean, and establish some sharp inequalities for this mean in terms of power and logarithmic means. From these a nice chain of inequalities involving Gauss compound mean, Toader mean and Toader-Qi mean is presented.

1. INTRODUCTION

Let the function $p : (0, \infty) \rightarrow \mathbb{R}$ be strictly monotone and let $n \in \mathbb{R}$. The Toader' family of mean values is defined in [1] by

$$M_{p,n}(a, b) = p^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} p(r_n(\theta)) d\theta \right),$$

where

$$r_n(\theta) = \begin{cases} (a^n \cos^2 \theta + b^n \sin^2 \theta)^{1/n} & \text{if } n \neq 0, \\ a^{\cos^2 \theta} b^{\sin^2 \theta} & \text{if } n = 0 \end{cases}$$

for $\theta \in (0, 2\pi)$, p^{-1} is the inverse of a strictly monotonic function p . Also, it is obvious that

$$M_{p,n}(a, b) = p^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} p(r_n(\theta)) d\theta \right) = p^{-1} \left(\frac{2}{\pi} \int_0^{\pi/2} p(r_n(\theta)) d\theta \right).$$

When $p(x) = 1/x$ and $n = 2$, we see that

$$(1.1) \quad M_{1/x,2}(a, b) = \frac{\pi/2}{\int_0^{\pi/2} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{-1/2} d\theta} = AGM(a, b)$$

is the classical Gauss compound mean related to the complete integrals of the first kind. Some inequalities involving AGM can be found in [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].

While letting $p(x) = x$ and $n = 2$ yields

$$(1.2) \quad M_{x,2}(a, b) = \frac{2}{\pi} \int_0^{\pi/2} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{1/2} d\theta = \mathcal{T}(a, b),$$

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which is the Toader mean related to the complete integrals of the second kind. There has some papers studied bounds for the mean in terms of other simpler means such as [8], [12], [13], [14], [15], [16], [17], [18], [19], [20].

Taking $p(x) = x^q$ ($q \neq 0$) and $n = 0$ gives

$$M_{x^q,0}(a,b) = \left(\frac{2}{\pi} \int_0^{\pi/2} a^{q \cos^2 \theta} b^{q \sin^2 \theta} d\theta \right)^{1/q}.$$

In particular, we have

$$(1.3) \quad M_{x,0}(a,b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta.$$

The mean $M_{x^q,0}(a,b)$ seems to be mysterious so that the author said that he did not know how to determine any mean at the end of the paper [1]. Very recently, Qi et al. [21, Lemma 2.1] revealed the surprising relation between the mean $M_{x^q,0}(a,b)$ and modified Bessel functions of the first kind. They proved that

Theorem 1 ([21, Lemma 2.1]). *For positive numbers $a, b > 0$, we have*

$$(1.4) \quad M_{x,0}(a,b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta = \sqrt{ab} I_0 \left(\ln \sqrt{\frac{a}{b}} \right)$$

and

$$(1.5) \quad M_{x^q,0}(a,b) = \left(\frac{2}{\pi} \int_0^{\pi/2} a^{q \cos^2 \theta} b^{q \sin^2 \theta} d\theta \right)^{1/q} = \sqrt{ab} I_0^{1/q} \left(q \ln \sqrt{\frac{a}{b}} \right),$$

where

$$I_v(z) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(v+n+1)} \left(\frac{z}{2} \right)^{2n+v}, \quad z \in \mathbb{C}, \quad v \in \mathbb{R} \setminus \{-1, -2, \dots\}$$

denotes the modified Bessel functions of the first kind and

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{-1, -2, \dots\}$$

is the classical gamma function.

Remark 1. Let $M(a,b)$ be a homogeneous mean of positive arguments a and b . Then

$$M(a,b) = \sqrt{ab} M(e^t, e^{-t}),$$

where $t = (1/2) \ln(b/a)$.

Since the mean $M_{x^q,0}(a,b)$ is symmetric and homogeneous with respect to a and b , we can assume that $b > a > 0$ and let $t = (1/2) \ln(b/a) > 0$. Then by Remark 1 Qi's relations (1.4) and (1.5) can be rewritten as

$$(1.6) \quad \frac{M_{x,0}(a,b)}{\sqrt{ab}} = \frac{2}{\pi} \int_0^{\pi/2} e^{t \cos 2\theta} d\theta = I_0(t)$$

and

$$\frac{M_{x^q,0}(a,b)}{\sqrt{ab}} = \left(\frac{2}{\pi} \int_0^{\pi/2} e^{qt \cos 2\theta} d\theta \right)^{1/q} = I_0^{1/q}(qt).$$

Also, from (1.6) it is easy to verify that

$$(1.7) \quad I_0(t) = \frac{2}{\pi} \int_0^{\pi/2} \cosh(t \cos \theta) d\theta = \frac{2}{\pi} \int_0^{\pi/2} \cosh(t \sin \theta) d\theta$$

(see also [22, p. 376, 9.6.16]).

Similarly, the logarithmic mean, identric (exponential) mean and power mean of order p defined by

$$\begin{aligned} L(a, b) &= \frac{b-a}{\ln b - \ln a}, \quad \mathcal{I}(a, b) = e^{-1} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, \\ A_p(a, b) &= \left(\frac{a^p + b^p}{2} \right)^{1/p} \text{ if } p \neq 0 \text{ and } A_0(a, b) = \sqrt{ab} \end{aligned}$$

can be rewritten as

$$\begin{aligned} \frac{L(a, b)}{\sqrt{ab}} &= \frac{\sinh t}{t}, \quad \frac{\mathcal{I}(a, b)}{\sqrt{ab}} = \exp \left(\frac{t}{\tanh t} - 1 \right), \\ \frac{A_p(a, b)}{\sqrt{ab}} &= \cosh pt \text{ if } p \neq 0 \text{ and } \frac{A_0(a, b)}{\sqrt{ab}} = 1. \end{aligned}$$

In particular, the arithmetic and geometric means $A = A_1(a, b)$ and $G(a, b) = A_0(a, b) = \sqrt{ab}$ can be changed into $A(a, b)/\sqrt{ab} = \cosh t$ and $G(a, b)/\sqrt{ab} = 1$, respectively.

Further, Qi et al. showed that

Theorem 2 ([21, Theorem 1.1]). *The double inequality*

$$(1.8) \quad L(a, b) < \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta < \mathcal{I}(a, b)$$

holds for $a, b > 0$ with $a \neq b$. Consequently,

$$L(a^q, b^q)^{1/q} < \left(\frac{2}{\pi} \int_0^{\pi/2} a^{q \cos^2 \theta} b^{q \sin^2 \theta} d\theta \right)^{1/q} < \mathcal{I}(a^q, b^q)^{1/q}.$$

Remark 2. Due to that Qi et al. first revealed the surprising connection between the mean $M_{x,0}(a, b)$ and the modified Bessel functions of the first kind, and established inequalities for the mean in terms of logarithmic and identric (exponential) means, we call the mean $M_{x,0}(a, b)$ defined by (1.3) Toader-Qi mean and denote by $TQ(a, b)$.

At the end of paper [21], Qi et al. gave some improvements for the second inequality in (1.8), that is,

$$(1.9) \quad \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta < \frac{A(a, b) + G(a, b)}{2} < \frac{2A(a, b) + G(a, b)}{3} < \mathcal{I}(a, b).$$

The aim of this paper is to present some sharp inequalities for the Toader-Qi mean $TQ(a, b)$, or equivalently, modified Bessel functions of the first kind $I_0(t)$ in terms of hyperbolic functions.

2. LEMMAS

To formulate properties of the Toader-Qi mean $TQ(a, b)$ or $I_0(t)$ and our results, we need some lemmas.

Lemma 1 ([23, Problem 32]). *Let $\binom{n}{k}$ be the number of combinations of n objects taken k at a time, that is,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then we have

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

Lemma 2 ([23, Problems 85, 94]). *The two given sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ satisfy the conditions*

$$b_n > 0; \quad \sum_{n=0}^{\infty} b_n t^n \text{ converges for all values of } t; \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = s.$$

Then $\sum_{n=0}^{\infty} a_n t^n$ converges too for all values of t and in addition

$$\lim_{t \rightarrow \infty} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s.$$

Lemma 3. *The Wallis ratio W_n defined by*

$$(2.1) \quad W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n} n!^2} = \frac{\Gamma(n+1/2)}{\Gamma(1/2) \Gamma(n+1)}$$

is strictly decreasing and log-convex for all integers $n \geq 0$.

Proof. Direct computations gives

$$(2.2) \quad \begin{aligned} \frac{W_{n+1}}{W_n} &= \frac{2n+1}{2n+2} = 1 - \frac{1}{2n+2} < 1, \\ \frac{W_{n+2}}{W_{n+1}} \bigg/ \frac{W_{n+1}}{W_n} &= \frac{2n+3}{2n+4} \frac{2n+2}{2n+1} = 1 + \frac{1}{(n+2)(2n+1)} > 1, \end{aligned}$$

which proves the lemma. □

Lemma 4 ([24], [25, (2.8)]). *For all $x > 0$ and all $a \in (0, 1)$, it holds that*

$$(2.3) \quad \frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}}.$$

Lemma 5. *The sequence $\{s_n\}_{n \geq 0}$ defined by*

$$(2.4) \quad s_n = \frac{(2n)!(2n+1)!}{2^{4n} n!^4}$$

is strictly decreasing, and $\lim_{n \rightarrow \infty} s_n = 2/\pi$.

Proof. An easy computation yields

$$(2.5) \quad \frac{s_{n+1}}{s_n} = \frac{(2n+2)!(2n+3)!}{2^{4n+4} (n+1)!^4} \bigg/ \frac{(2n)!(2n+1)!}{2^{4n} n!^4} = \frac{1}{4} \frac{(2n+1)(2n+3)}{(n+1)^2} < 1,$$

which shows that the sequence $\{s_n\}$ is strictly decreasing for all $n \geq 0$. To calculate $\lim_{n \rightarrow \infty} s_n$, we write s_n as

$$s_n = \frac{(2n)!(2n+1)!}{2^{4n}n!^4} = (2n+1) \left(\frac{(2n-1)!!}{2^n n!} \right)^2 = \frac{2n+1}{\Gamma(1/2)^2} \left(\frac{\Gamma(n+1/2)}{\Gamma(n+1)} \right)^2,$$

and it suffices to prove that

$$\lim_{n \rightarrow \infty} \frac{(n+1/2)\Gamma(n+1/2)^2}{\Gamma(n+1)^2} = 1.$$

Making use of Lemma 4 yields

$$1 = \frac{n+1/2}{n+1/2} < \frac{(n+1/2)\Gamma(n+1/2)^2}{\Gamma(n+1)^2} < \frac{n+1/2}{n},$$

which implies the desired assertion. \square

Lemma 6 ([26]). *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r, r)$ ($r > 0$) with $b_k > 0$ for all k . If the sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k , then the function $t \mapsto A(t)/B(t)$ is also increasing (decreasing) on $(0, r)$.*

Lemma 7 ([27, Corollary 2.3.]). *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on \mathbb{R} with $b_k > 0$ for all k . If for certain $m \in \mathbb{N}$, the non-constant sequence $\{a_k/b_k\}$ is increasing (decreasing) for $0 \leq k \leq m$ and decreasing (increasing) for $k \geq m$, then there is a unique $t_0 \in (0, \infty)$ such that the function A/B is increasing (decreasing) on $(0, t_0)$ and decreasing (increasing) on (t_0, ∞) .*

3. PROPERTIES

Now we give some simple properties of the Toader-Qi mean $TQ(a, b)$ or $I_0(t)$.

By the identities (1.7), the following property is immediate.

Property 1. *For $t > 0$, it holds that*

$$1 < I_0(t) < \cosh t,$$

or equivalently, the double inequality

$$(3.1) \quad \sqrt{ab} < TQ(a, b) < \frac{a+b}{2}$$

holds for $a, b > 0$ with $a \neq b$.

Making a change of variable $\sin \theta = x$ in the second identity of (1.7) yields

Property 2. *We have*

$$(3.2) \quad I_0(t) = \frac{2}{\pi} \int_0^1 \frac{\cosh(tx)}{\sqrt{1-x^2}} dx$$

(see also [22, p. 376, 9.6.18]).

Property 3. *For $t > 0$, it holds that*

$$(3.3) \quad \frac{e^t}{1+2t} < I_0(t) < \frac{e^t}{\sqrt{1+2t}},$$

or equivalently,

$$(3.4) \quad \frac{b}{1 + \ln(b/a)} < TQ(a, b) < \frac{b}{\sqrt{1 + \ln(b/a)}}$$

holds for $b > a > 0$. Consequently, we have

$$(3.5) \quad \lim_{t \rightarrow \infty} e^{-t} I_0(t) = 0$$

or

$$(3.6) \quad \lim_{x \rightarrow 0^+} TQ(x, 1) = 0.$$

Proof. An easy verification shows that both the functions $1/\sqrt{1-x^2}$ and $\cosh(tx)$ are increasing with respect to x on $[0, 1]$. Using the Chebyshev integral inequality to the formula (3.2) we get

$$I_0(t) > \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \cosh(tx) dx = \frac{\sinh t}{t} = \frac{e^t}{2t} \left(1 - \frac{1}{e^{2t}}\right) > \frac{e^t}{2t+1},$$

where the last inequality holds due to $e^{2t} > 1 + 2t$, which proves the first inequality in (3.3).

On the other hand, we have

$$\begin{aligned} e^{-t} I_0(t) &= \frac{2}{\pi} \int_0^{\pi/2} e^{-t} e^{t \cos 2\theta} d\theta = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{e^{2t \sin^2 \theta}} < \int_0^{\pi/2} \frac{d\theta}{1 + 2t \sin^2 \theta} \\ &= \frac{2}{\pi} \left[\frac{\arctan(\sqrt{1+2t} \tan \theta)}{\sqrt{1+2t}} \right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{\sqrt{1+2t}}, \end{aligned}$$

which proves the second inequality in (3.3). From the double inequality (3.3) the limit relation (3.5) easily follows.

Substituting $t = (1/2) \ln(b/a)$ ($b > a > 0$) into (3.3) and (3.5) gives (3.4) and (3.6). \square

Remark 3. By the limit relation (3.6) and homogeneous of $TQ(a, b)$ with respect to positive numbers a and b , we see that the Toader- Q_i mean $TQ(a, b)$ can be extended continuously to the domain $\{(a, b) | a, b \geq 0\}$.

Property 4. We have

$$(3.7) \quad I_0(t)^2 = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n}.$$

Proof. By Cauchy product formula and Lemma 1, it is obtained that

$$\begin{aligned} I_0(t)^2 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{1}{2^{2k} k!^2} \frac{1}{2^{2(n-k)} (n-k)!^2} \right) t^{2n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2^{2n} n!^2} \sum_{k=0}^n \frac{n!^2}{k!^2 (n-k)!^2} \right) t^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n}. \end{aligned}$$

\square

Property 5. *The function*

$$t \mapsto I_0(t) \left/ \sqrt{\frac{\sinh 2t}{2t}} \right.$$

is strictly decreasing from $(0, \infty)$ onto $(\sqrt{2/\pi}, 1)$. Consequently, the double inequality

$$(3.8) \quad \sqrt{\frac{\sinh 2t}{\pi t}} < I_0(t) < \sqrt{\frac{\sinh 2t}{2t}}$$

holds for $t > 0$, or equivalently, the double inequality

$$(3.9) \quad \sqrt{\frac{2}{\pi}} \sqrt{L(a, b) A(a, b)} < TQ(a, b) < \sqrt{L(a, b) A(a, b)}$$

holds for $a, b > 0$ with $a \neq b$, where $\sqrt{2/\pi}$ and 1 are the best possible.

Proof. Using the identity (3.7) we have

$$R_0(t) := \frac{I_0(t)^2}{(\sinh 2t)/(2t)} = \frac{\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n}}{\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} t^{2n}} := \frac{\sum_{n=0}^{\infty} a_n t^{2n}}{\sum_{n=0}^{\infty} b_n t^{2n}}.$$

It is obvious that

$$\frac{a_n}{b_n} = \frac{(2n)!}{2^{2n} n!^4} \left/ \frac{2^{2n}}{(2n+1)!} \right. = \frac{(2n)! (2n+1)!}{2^{4n} n!^4} = s_n.$$

By Lemma 5 it follows that the sequence $\{a_n/b_n\}$ is strictly decreasing for all integers $n \geq 0$, so is the function R_0 on $(0, \infty)$ by Lemma 6. Consequently, it is obtained that

$$\frac{2}{\pi} = \lim_{t \rightarrow \infty} R_0(t) < R_0(t) = \lim_{t \rightarrow 0} R_0(t) = 1,$$

where the first equality holds due to

$$(3.10) \quad \lim_{t \rightarrow \infty} R_0(t) = \lim_{t \rightarrow \infty} \frac{\sum_{n=0}^{\infty} a_n t^{2n}}{\sum_{n=0}^{\infty} b_n t^{2n}} = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(2n)! (2n+1)!}{2^{4n} n!^4} = \frac{2}{\pi}$$

by Lemmas 2 and 5.

This completes the proof. \square

Remark 4. *The limit relation 3.10 implies that*

$$\lim_{t \rightarrow \infty} \sqrt{t} e^{-t} I_0(t) = \frac{1}{\sqrt{2\pi}} \text{ or } I_0(t) \sim \frac{e^t}{\sqrt{2\pi t}} \text{ as } t \rightarrow \infty$$

(see also [22, p. 377, 9.7.1]).

4. MAIN RESULTS

Due to Remark 1, almost all of inequalities for homogeneous symmetric bivariate means can be transformed equivalently into the corresponding ones for hyperbolic functions and vice versa, for example, the double inequalities (3.8) and (3.9) are equivalent to each other. Therefore, for convenience, we only present sharp inequalities for the modified Bessel functions of the first kind $I_0(t)$ in terms of hyperbolic functions in this section.

Theorem 3. *The double inequality*

$$(4.1) \quad \sqrt{(\lambda \cosh t + 1 - \lambda) \frac{\sinh t}{t}} < I_0(t) < \sqrt{(\delta \cosh t + 1 - \delta) \frac{\sinh t}{t}}$$

holds for all $t > 0$ if and only $\lambda \in [0, 2/\pi]$ and $\delta \in [\delta_0, \infty)$, where $\delta_0 \approx 0.67664$ is defined by

$$\delta_0 = \frac{t_0 I_0(t_0)^2 - \sinh t_0}{(\cosh t_0 - 1) \sinh t_0},$$

here t_0 is the unique solution of the equation

$$\frac{d}{dt} \left(\frac{t I_0(t)^2 - \sinh t}{(\cosh t - 1) \sinh t} \right) = 0$$

on $(0, \infty)$.

Proof. Let us consider the ratio

$$R_1(t) = \frac{I_0(t)^2 - (\sinh t)/t}{(\cosh t - 1)(\sinh t)/t} = \frac{\sum_{n=1}^{\infty} \left(\frac{(2n)!}{2^{2n} n!^4} - \frac{1}{(2n+1)!} \right) t^{2n}}{\sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n+1)!} t^{2n}} := \frac{\sum_{n=1}^{\infty} c_n t^{2n}}{\sum_{n=1}^{\infty} d_n t^{2n}}.$$

To determine the monotonicity of $R_2(t)$, it suffices to observe the monotonicity of the sequence $\{c_n/d_n\}$. We have

$$\begin{aligned} \frac{c_n}{d_n} &= \left(\frac{(2n)!}{2^{2n} n!^4} - \frac{1}{(2n+1)!} \right) \bigg/ \frac{2^{2n}-1}{(2n+1)!} \\ &= \left(2^{2n} \times \frac{(2n)!(2n+1)!}{2^{4n} n!^4} - 1 \right) \bigg/ (2^{2n}-1) = \frac{2^{2n} s_n - 1}{2^{2n} - 1}, \end{aligned}$$

where s_n is defined by (2.4). Then it is obtained by the relation (2.5) that

$$\begin{aligned} \frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} &= \frac{2^{2n+2} s_{n+1} - 1}{2^{2n+2} - 1} - \frac{2^{2n} s_n - 1}{2^{2n} - 1} \\ &= \frac{2^{2n+2} \frac{(2n+1)(2n+3)}{4(n+1)^2} s_n - 1}{2^{2n+2} - 1} - \frac{2^{2n} s_n - 1}{2^{2n} - 1} \\ &= -\frac{2^{2n} \times s'_n}{(n+1)^2 (2^{2n+2} - 1) (2^{2n} - 1)}, \end{aligned}$$

where

$$s'_n = (2^{2n} + 3n^2 + 6n + 2) s_n - (3n^2 + 6n + 3).$$

Since the sequence $\{s_n\}$ is strictly decreasing for $n \geq 0$ and $\lim_{n \rightarrow \infty} s_n = 2/\pi > 3/5$, we get

$$\begin{aligned} s'_n &> (2^{2n} + 3n^2 + 6n + 2) \frac{3}{5} - (3n^2 + 6n + 3) \\ &= \frac{3}{5} (2^{2n} - (2n^2 + 4n + 3)) > 0 \end{aligned}$$

for $n \geq 3$. Therefore, the sequence $\{c_n/d_n\}$ is strictly decreasing for $n \geq 3$.

On the other hand, a direct computation yields

$$\frac{c_1}{d_1} = \frac{2}{3} < \frac{c_2}{d_2} = \frac{41}{60} > \frac{c_3}{d_3} = \frac{19}{28}.$$

These shows that the sequence $\{c_n/d_n\}$ is strictly increasing for $n = 1, 2$ and decreasing for $n \geq 2$. By Lemma 7, there is a unique $t_0 \in (0, \infty)$ such that the function R_2 is strictly increasing on $(0, t_0)$ and decreasing on (t_0, ∞) . Therefore, we conclude that

$$\frac{2}{\pi} = \min(R_1(0^+), R_1(\infty)) < R_1(t) \leq R_1(t_0) = \delta_0,$$

where the first equality holds due to $R_1(0^+) = 2/3$ and

$$R_1(\infty) = \lim_{t \rightarrow \infty} \frac{\sum_{n=1}^{\infty} c_n t^{2n}}{\sum_{n=1}^{\infty} d_n t^{2n}} = \lim_{n \rightarrow \infty} \frac{c_n}{d_n} = \lim_{n \rightarrow \infty} \frac{2^{2n} s_n - 1}{2^{2n} - 1} = \frac{2}{\pi}$$

by Lemmas 2 and 5. Solving the equation $R_1'(t) = 0$, we find that $t_0 \approx 2.7113555314$, and $R_1(t_0) \approx 0.67664$.

Thus we complete the proof. \square

Theorem 4. *Let $p, q \in \mathbb{R}$. The double inequality*

$$(4.2) \quad (\cosh t)^{1-p} \left(\frac{\sinh t}{t} \right)^p < I_0(t) < q \frac{\sinh t}{t} + (1-q) \cosh t$$

holds for $t > 0$ if and only if $p \geq 3/4$ and $q \leq 3/4$.

Proof. (i) The necessity of the first inequality in (4.2) follows from the expansion in power series

$$I_0(t) - (\cosh t)^{1-p} \left(\frac{\sinh t}{t} \right)^p = \frac{1}{3} t^2 \left(p - \frac{3}{4} \right) + O(t^4).$$

Since the function $p \mapsto (\cosh t)^{1-p} ((\sinh t)/t)^p$ is decreasing, to prove the sufficiency, it is enough to prove that the first inequality in (4.2) holds for $p = 3/4$, that is,

$$I_0(t) > (\cosh t)^{1/4} \left(\frac{\sinh t}{t} \right)^{3/4},$$

which is equivalent to

$$I_0(t)^4 > \left(\frac{\sinh t}{t} \right)^3 \cosh t.$$

Expanding in power series yields

$$(\cosh t) \left(\frac{\sinh t}{t} \right)^3 = \sum_{n=0}^{\infty} \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!} t^{2n}.$$

On the other hand, by Cauchy product formula and formula (3.7), it is obtained that

$$I_0(t)^4 = \sum_{n=0}^{\infty} \sum_{k=0}^n \left(\frac{(2k)!}{2^{2k} k!^4} \frac{(2(n-k))!}{2^{2(n-k)} (n-k)!^4} \right) t^{2n} := \sum_{n=0}^{\infty} \sum_{k=0}^n u_{n,k} t^{2n}.$$

Thus it suffices to prove that

$$v_n = \sum_{k=0}^n u_{n,k} - \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!} \geq 0$$

for $n \geq 0$. To this end, we use the identity (2.1) to $u_{n,k}$, then apply Lemma 3 and inequality (2.3), to get that

$$\begin{aligned}
u_{n,k} &= \frac{1}{k!^2 (n-k)!^2} \frac{(2k)!}{2^{2k} k!^2} \frac{(2(n-k))!}{2^{2(n-k)} (n-k)!^2} \\
&= \frac{1}{k!^2 (n-k)!^2} W_k W_{n-k} \\
&> \frac{1}{k!^2 (n-k)!^2} W_{n/2}^2 = \frac{1}{k!^2 (n-k)!^2} \left(\frac{\Gamma(n/2 + 1/2)}{\Gamma(1/2) \Gamma(n/2 + 1)} \right)^2 \\
&> \frac{1}{\pi} \frac{1}{k!^2 (n-k)!^2} \frac{1}{(n/2 + 1/2)}.
\end{aligned}$$

Then it follows from Lemma 1 that

$$\begin{aligned}
\sum_{k=0}^n u_{n,k} &> \sum_{k=0}^n \frac{1}{\pi} \frac{1}{k!^2 (n-k)!^2} \frac{1}{(n/2 + 1/2)} \\
&= \frac{2}{\pi (n+1) n!^2} \sum_{k=0}^n \frac{n!^2}{k!^2 (n-k)!^2} = \frac{2}{\pi (n+1) n!^2} \frac{(2n)!}{n!^2},
\end{aligned}$$

and then,

$$\begin{aligned}
v_n &= \sum_{k=0}^n u_{n,k} - \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!} > \frac{2}{\pi (n+1) n!^2} \frac{(2n)!}{n!^2} - \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!} \\
&= \frac{1}{\pi} \frac{2^{2n+1} (2^{2n+2} - 1)}{(2n+3)!} \left(\frac{2^{2n+1} (2n+3) (2n)! (2n+1)!}{2^{2n+2} - 1} \frac{1}{2^{4n} n!^4} - \pi \right) \\
&= \frac{1}{\pi} \frac{2^{2n+1} (2^{2n+2} - 1)}{(2n+3)!} \left(\frac{2^{2n+2}}{2^{2n+2} - 1} \left(n + \frac{3}{2} \right) s_n - \pi \right),
\end{aligned}$$

where s_n is defined by (2.4).

By Lemma 5 it follows that

$$\frac{2^{2n+2}}{2^{2n+2} - 1} \left(n + \frac{3}{2} \right) s_n - \pi > \left(n + \frac{3}{2} \right) \frac{2}{\pi} - \pi > 0$$

for $n \geq 4$, which implies that $v_n > 0$ for $n \geq 4$.

This together with the facts that $v_0 = v_1 = 0$, $v_2 = 3/80$, $v_3 = 4/189$ indicates that $v_n \geq 0$ for all integers $n \geq 0$, which proves the sufficiency.

(ii) The necessity of the second inequality in (4.2) can be derived from the expansion in power series

$$I_0(t) - q \frac{\sinh t}{t} - (1-q) \cosh t = \frac{1}{3} t^2 \left(q - \frac{3}{4} \right) + O(t^4).$$

To prove the sufficiency, let us consider the ratio

$$\begin{aligned}
R_2(t) &= \frac{\cosh t - I_0(t)}{\cosh t - (\sinh t)/t} = \frac{\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n} n!^2}}{\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+1)!}} \\
&= \frac{\sum_{n=0}^{\infty} \frac{2^n n! - (2n-1)!!}{2^n n! (2n)!} t^{2n}}{\sum_{n=0}^{\infty} \frac{2n}{(2n+1)!} t^{2n}} := \frac{\sum_{n=1}^{\infty} \alpha_n t^{2n}}{\sum_{n=1}^{\infty} \beta_n t^{2n}}.
\end{aligned}$$

Simplifying yields

$$\frac{\alpha_n}{\beta_n} = \frac{(2n+1)(2^n n! - (2n-1)!!)}{n2^{n+1}n!} = \frac{2n+1}{2n}(1 - W_n),$$

where W_n is defined by (2.1). Using the recursive relation (2.2) we have

$$\begin{aligned} \frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n} &= \frac{2n+3}{2n+2}(1 - W_{n+1}) - \frac{2n+1}{2n}(1 - W_n) \\ &= \frac{2n+3}{2n+2} \left(1 - \frac{2n+1}{2n+2}W_n\right) - \frac{2n+1}{2n}(1 - W_n) \\ &= \frac{1}{2n(n+1)} \left(\frac{(n+2)(2n+1)}{2(n+1)}W_n - 1\right) := \frac{\gamma_n - 1}{2n(n+1)}. \end{aligned}$$

Since

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{(2n+3)(n+3)}{2(n+2)^2} = 1 + \frac{n+1}{2(n+2)^2} > 1,$$

the sequence $\{\gamma_n\}$ is strictly increasing for $n \geq 1$, we have $\gamma_n \geq \gamma_1 = 9/8 > 1$. This in turn implies that the sequence $\{\alpha_n/\beta_n\}$ is strictly increasing for $n \geq 1$, so that the ratio R_1 is increasing for $t > 0$. Thus we conclude that

$$R_2(t) = \frac{\cosh t - I_0(t)}{\cosh t - (\sinh t)/t} > \lim_{t \rightarrow 0^+} \frac{\cosh t - I_0(t)}{\cosh t - (\sinh t)/t} = \frac{3}{4},$$

which proves the sufficiency.

This completes the proof. \square

Theorem 5. Let $p \in (0, \infty)$. Then (i) the double inequality

$$(4.3) \quad 1 - \frac{1}{2p^2} + \frac{1}{2p^2} \cosh pt < I_0(t) < 1 - \frac{1}{2q^2} + \frac{1}{2q^2} \cosh qt$$

holds for $t > 0$ if and only if $p \in (0, \sqrt{3}/2]$ and $q \in [1, \infty)$.

(ii) For $p \in (\sqrt{3}/2, 1)$, the inequality

$$(4.4) \quad I_0(t) \geq 1 - \frac{\lambda_0}{p^2} + \frac{\lambda_0}{p^2} \cosh pt$$

holds for $t > 0$ with

$$\lambda_0 = \frac{I_0(t_0) - 1}{(\cosh pt_0 - 1)/p^2},$$

where t_0 is the unique solution of the equation

$$\frac{d}{dt} \frac{I_0(t) - 1}{\cosh pt - 1} = 0$$

on $(0, \infty)$

Proof. (i) To prove the necessity for the inequalities (4.3) to hold, let us consider the ratio

$$R_3(t) = \frac{I_0(t) - 1}{(\cosh pt - 1)/p^2} = \frac{\sum_{n=1}^{\infty} \frac{t^{2n}}{2^{2n}n!^2}}{\sum_{n=1}^{\infty} \frac{p^{2n-2}t^{2n}}{(2n)!}} := \frac{\sum_{n=1}^{\infty} \mu_n t^{2n}}{\sum_{n=1}^{\infty} \nu_n t^{2n}}.$$

A simple computation yields

$$\begin{aligned} \frac{\mu_{n+1}}{\nu_{n+1}} - \frac{\mu_n}{\nu_n} &= \frac{1}{p^{2n}} \frac{(2n+2)!}{2^{2n+2} (n+1)!^2} - \frac{1}{p^{2n-2}} \frac{(2n)!}{2^{2n} n!^2} \\ &= -\frac{1}{p^{2n}} \frac{(2n)!}{2^{2n} n!^2} \left(p^2 - \frac{2n+1}{2n+2} \right) \\ &\begin{cases} \leq 0 & \text{if } p^2 \geq \max_{n \in \mathbb{N}} \frac{2n+1}{2n+2} = 1, \\ \geq 0 & \text{if } p^2 \leq \min_{n \in \mathbb{N}} \frac{2n+1}{2n+2} = \frac{3}{4}. \end{cases} \end{aligned}$$

These show that the sequence $\{\mu_n/\nu_n\}$ is strictly decreasing for $n \geq 1$ if $p \geq 1$ and increasing if $0 < p \leq \sqrt{3}/2$, and so is R_3 by Lemma 6. Hence, we get that

$$\begin{aligned} R_3(t) &< \lim_{t \rightarrow 0^+} R_3(t) = \frac{1}{2} \text{ if } p \geq 1, \\ R_3(t) &> \lim_{t \rightarrow 0^+} R_3(t) = \frac{1}{2} \text{ if } 0 < p \leq \sqrt{3}/2, \end{aligned}$$

which proves (4.3).

The necessity for the first inequality in (4.3) to hold follows from

$$\lim_{t \rightarrow 0^+} \frac{I_0(t) - \left(1 - \frac{1}{2p^2} + \frac{1}{2p^2} \cosh pt\right)}{t^4} = -\frac{1}{48} \left(p^2 - \frac{3}{4}\right) \geq 0.$$

We prove the necessity for the second inequality in (4.3) to hold by proof by contradiction. Assume that there is a $q_0 \in (\sqrt{3}/2, 1)$ such that the second inequality in (4.3) holds for $t > 0$. Then there must be

$$\lim_{t \rightarrow \infty} \frac{I_0(t) - \left(1 - \frac{1}{2q_0^2} + \frac{1}{2q_0^2} \cosh q_0 t\right)}{e^{q_0 t}} \leq 0.$$

Making use of the first inequality in (3.3) leads to

$$\frac{I_0(t) - \left(1 - \frac{1}{2q_0^2} + \frac{1}{2q_0^2} \cosh q_0 t\right)}{e^{q_0 t}} > \frac{e^{(1-q_0)t}}{1+2t} - \left(1 - \frac{1}{2q_0^2}\right) e^{-q_0 t} + \frac{1}{2q_0^2} \frac{1+e^{-2q_0 t}}{2} \rightarrow \infty$$

as $t \rightarrow \infty$, which gives a contradiction.

(ii) When $p \in (\sqrt{3}/2, 1)$, it is clear that there exist a $n_0 > 1$ such that $\mu_{n+1}/\nu_{n+1} - \mu_n/\nu_n \leq 0$ for $1 \leq n \leq n_0$ and $\mu_{n+1}/\nu_{n+1} - \mu_n/\nu_n \geq 0$ for $n \geq n_0$. By Lemma 7 it follows that there is a $t_0 > 0$ such that R_3 is decreasing on $(0, t_0)$ and increasing on (t_0, ∞) , and therefore, we have

$$R_3(t) \geq R_3(t_0) = \lambda_0,$$

that is, the inequality (4.4). \square

Similar to [28, Remark 2.] that the function

$$p \mapsto 1 - \frac{1}{2p^2} + \frac{1}{2p^2} \cosh pt$$

is increasing on $(0, \infty)$. Letting $p = \sqrt{3}/2, 3/4, 1/\sqrt{2}, 2/3, 1/2$ and $q = 1$ in Theorem 5, we have

Corollary 1. *It holds that*

$$\begin{aligned} \sqrt{\cosh t} &< 2 \cosh \frac{t}{2} - 1 < \frac{9}{8} \cosh \frac{2}{3} - \frac{1}{8} < \cosh \frac{t}{\sqrt{2}} < \\ \frac{8}{9} \cosh \frac{3t}{4} + \frac{1}{9} &< \frac{2}{3} \cosh \frac{\sqrt{3}t}{2} + \frac{1}{3} < I_0(t) < \frac{1 + \cosh t}{2} \end{aligned}$$

for $t > 0$.

Proof. It remains to be proved that

$$\sqrt{\cosh t} < 2 \cosh \frac{t}{2} - 1,$$

which follows from

$$\left(2 \cosh \frac{t}{2} - 1\right)^2 - \cosh t = 2 \left(\cosh \frac{t}{2} - 1\right)^2 > 0.$$

□

Theorem 6. *For $p > 0$, the inequality*

$$(4.5) \quad I_0(t) > (\cosh pt)^{1/(2p^2)}$$

holds for $t > 0$ if and only if $p \geq \sqrt{6}/4 \approx 0.61237$. Its reverse holds if and only if $p \in (0, 1/2]$. In particular, we have

$$(4.6) \quad \sqrt{\cosh t} < \cosh \frac{t}{\sqrt{2}} < \left(\cosh \frac{\sqrt{6}t}{4}\right)^{4/3} < I_0(t) < \left(\cosh \frac{t}{2}\right)^2 < e^{t^2/4}$$

Proof. (i) The necessary condition for the inequality (4.5) to hold follows from

$$\lim_{t \rightarrow 0} \frac{I_0(t) - (\cosh pt)^{1/(2p^2)}}{t^4} = \frac{1}{24} \left(p^2 - \frac{3}{8}\right) \geq 0.$$

Since the function $p \mapsto (\cosh pt)^{1/(2p^2)}$ is strictly decreasing which is proved in [29, Lemma 2], to prove the sufficiency, it suffices to prove that the inequality (4.5) holds for $t > 0$ when $p = \sqrt{6}/4$. In fact, utilizing Theorem 5, we only need to prove that

$$\frac{2}{3} \cosh \frac{\sqrt{3}t}{2} + \frac{1}{3} > \left(\cosh \frac{\sqrt{6}t}{4}\right)^{4/3},$$

which is equivalent to

$$f_1(x) := \ln \left(\frac{2}{3} \cosh(\sqrt{2}x) + \frac{1}{3} \right) - \frac{4}{3} \ln(\cosh x) > 0$$

for $x > 0$, where $x = \sqrt{6}t/4$.

Differentiation gives

$$f_1'(x) = \frac{1}{3} \frac{f_2(x)}{(2 \cosh(\sqrt{2}x) + 1) \cosh x},$$

where

$$f_2(x) = 6\sqrt{2} \sinh(\sqrt{2}x) \cosh x - 8 \cosh(\sqrt{2}x) \sinh x - 4 \sinh x.$$

Employing product into sum formula and Taylor expansion yields

$$\begin{aligned}
f_2(x) &= (3\sqrt{2}-4) \sinh(\sqrt{2}+1)x + (3\sqrt{2}+4) \sinh(\sqrt{2}-1)x - 4 \sinh x \\
&= \sum_{n=1}^{\infty} \frac{(3\sqrt{2}-4)(\sqrt{2}+1)^{2n-1} + (3\sqrt{2}+4)(\sqrt{2}-1)^{2n-1} - 4}{(2n-1)!} x^{2n-1} \\
&: = \sum_{n=1}^{\infty} \frac{\xi_n}{(2n-1)!} x^{2n-1}.
\end{aligned}$$

Letting $\eta_n = (\sqrt{2}+1)^{2n-1}$ and noting that $(\sqrt{2}-1)^{2n-1} = 1/\eta_n$, we have

$$\begin{aligned}
\eta_n \xi_n &= (3\sqrt{2}-4) \eta_n^2 - 4\eta_n + (3\sqrt{2}+4) \\
&= (3\sqrt{2}-4) (\eta_n - \sqrt{2}-1) (\eta_n - 5\sqrt{2}-7) \\
&= (3\sqrt{2}-4) (\eta_n - \eta_1) (\eta_n - \eta_2).
\end{aligned}$$

It thus can be seen that $\xi_1 = \xi_2 = 0$ and $\xi_n > 0$ for $n \geq 3$ in view of $\eta_n > \eta_2 > \eta_1$, which proves $f_2(x) > 0$. Hence, $f'_1(x) > 0$, and then $f_1(x) > f_1(0) = 0$ for $x > 0$. Thus the sufficiency follows.

(ii) The sufficiency follows from the last inequality in Corollary 1 with the decreasing property of the function $p \mapsto (\cosh pt)^{1/(2p^2)}$ on $(0, \infty)$. It remains to be proved the necessity. If there is a $p_0 \in (1/2, \sqrt{6}/4)$ such that $I_0(t) < (\cosh p_0 t)^{1/(2p_0^2)}$ for $t > 0$, then there must be

$$\lim_{t \rightarrow \infty} \frac{I_0(t) - (\cosh p_0 t)^{1/(2p_0^2)}}{e^{t/(2p_0)}} \leq 0.$$

But by the first inequality in (3.3), we have

$$\frac{I_0(t) - (\cosh p_0 t)^{1/(2p_0^2)}}{e^{t/(2p_0)}} > \frac{1}{1+2t} \frac{e^t}{e^{t/(2p_0)}} - \left(\frac{1+e^{-2p_0 t}}{2} \right)^{1/(2p_0^2)} \rightarrow \infty$$

as $t \rightarrow \infty$, which yields a contradiction.

Taking $p = 1, 1/\sqrt{2}, \sqrt{6}/4, 1/2, 0^+$ gives the chain of inequalities (4.6).

The theorem is proved. \square

Theorem 7. Let $\theta \in [0, \pi/2]$. Then the inequality

$$(4.7) \quad I_0(t) > \frac{\cosh(t \cos \theta) + \cosh(t \sin \theta)}{2}$$

holds for $t > 0$ if and only if $\theta \in [\pi/8, 3\pi/8]$. In particular, it holds that

$$(4.8) \quad I_0(t) > \frac{1}{2} \left(\cosh \frac{\sqrt{2-\sqrt{2}}t}{2} + \cosh \frac{\sqrt{2+\sqrt{2}}t}{2} \right) > \frac{1}{2} \left(\cosh \frac{\sqrt{3}t}{2} + \cosh \frac{t}{2} \right) > \cosh \frac{t}{\sqrt{2}}$$

for $t > 0$.

Proof. The necessity can follow from

$$\lim_{t \rightarrow 0} \frac{I_0(t) - \frac{\cosh(t \cos \theta) + \cosh(t \sin \theta)}{2}}{t^4} = -\frac{1}{192} \cos 4\theta \leq 0,$$

which yields $4\theta \in [\pi/2, 3\pi/2]$, that is, $\theta \in [\pi/8, 3\pi/8]$.

Differentiation gives

$$\begin{aligned} \frac{\partial}{\partial \theta} \frac{\cosh(t \cos \theta) + \cosh(t \sin \theta)}{2} &= \frac{t^2 \sin 2\theta}{4} \left(\frac{\sinh(t \sin \theta)}{t \sin \theta} - \frac{\sinh(t \cos \theta)}{t \cos \theta} \right), \\ \frac{d}{dx} \frac{\sinh x}{x} &= \frac{1}{x} \left(\cosh x - \frac{\sinh x}{x} \right) > 0, \end{aligned}$$

which show that the function $\theta \mapsto [\cosh(t \cos \theta) + \cosh(t \sin \theta)]/2$ is decreasing on $[0, \pi/4]$ and increasing on $[\pi/4, \pi/2]$. Thus, to prove the sufficiency, it is enough to prove the inequality (4.7) holds when $\theta = \pi/8$.

Expanding in power series yields

$$R_4(t) := \frac{\cosh(t \cos \theta) + \cosh(t \sin \theta)}{2I_0(t)} = \frac{\sum_{n=0}^{\infty} \frac{\left(\frac{2-\sqrt{2}}{4}\right)^n + \left(\frac{2+\sqrt{2}}{4}\right)^n}{(2n)!} t^{2n}}{\sum_{n=0}^{\infty} \frac{2}{2^{2n} n!^2} t^{2n}} := \frac{\sum_{n=0}^{\infty} \rho_n t^{2n}}{\sum_{n=0}^{\infty} \sigma_n t^{2n}}.$$

Straightforward computations lead to

$$\begin{aligned} \frac{\rho_n}{\sigma_n} &= \frac{1}{2} \frac{n!^2 (\sqrt{2})^n}{(2n)!} \left((\sqrt{2}-1)^n + (\sqrt{2}+1)^n \right), \\ \frac{\rho_{n+1}}{\sigma_{n+1}} \bigg/ \frac{\rho_n}{\sigma_n} &= \frac{\sqrt{2}}{2} \frac{n+1}{2n+1} \frac{(\sqrt{2}-1)^{n+1} + (\sqrt{2}+1)^{n+1}}{(\sqrt{2}-1)^n + (\sqrt{2}+1)^n}, \\ \frac{\rho_{n+1}}{\sigma_{n+1}} \bigg/ \frac{\rho_n}{\sigma_n} - 1 &= -\frac{\sqrt{2}}{2(2n+1)} \frac{(n+\sqrt{2}-1)(\sqrt{2}-1)^{n-1} + (n-\sqrt{2}-1)(\sqrt{2}+1)^{n-1}}{(\sqrt{2}-1)^n + (\sqrt{2}+1)^n} < 0 \end{aligned}$$

for $n \geq 0$. This means that the sequence $\{\rho_n/\sigma_n\}$ is decreasing for $n \geq 0$, and so is R_4 , which proves the sufficiency.

Applying the decreasing property of the function $\theta \mapsto [\cosh(t \cos \theta) + \cosh(t \sin \theta)]/2$ and letting $\theta = \pi/8, \pi/6, \pi/4$ show the inequalities (4.8).

The proof of this theorem ends. \square

5. REMARKS

Remark 5. In the proof of Property 3, we in fact give a simple proof of the inequality $I_0(t) > (\sinh t)/t$, which is equivalent to the first inequality in (1.8). Furthermore, we have

$$(5.1) \quad I_0(t) > \frac{\sinh t}{t} + \frac{3(4-\pi)}{\pi} \frac{(t \sinh t - 2 \cosh t + 2)}{t^2}.$$

Indeed, Lupas [30] has proven that

$$T(f, g) \geq \frac{12}{(b-a)^4} \left(\int_a^b \left(x - \frac{a+b}{2} \right) f(x) dx \right) \left(\int_a^b \left(x - \frac{a+b}{2} \right) g(x) dx \right)$$

if both f, g are convex on interval $[a, b]$, where $T(f, g)$ is the Chebyshev functional defined by

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right).$$

Differentiation yields

$$\left(\frac{1}{\sqrt{1-x^2}} \right)'' = \frac{2x^2+1}{(1-x^2)^{5/2}} > 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2} \cosh(tx) = t^2 \cosh tx > 0.$$

Then by Lupas's inequality we have

$$\begin{aligned}
& \int_0^1 \frac{\cosh(tx)}{\sqrt{1-x^2}} dx - \left(\int_0^1 \frac{dx}{\sqrt{1-x^2}} \right) \left(\int_0^1 \cosh(tx) dx \right) \\
& > 12 \left(\int_0^1 \frac{x-1/2}{\sqrt{1-x^2}} dx \right) \left(\int_0^1 (x-1/2) \cosh(tx) dx \right) \\
& = \frac{3(4-\pi)(t \sinh t - 2 \cosh t + 2)}{2t^2},
\end{aligned}$$

which together with (3.2) implies (5.1).

When $0 < t < 0.8305\dots$, the lower bound given in (5.1) is weaker than one in [31, (1.5)]; when $t > 0.8305\dots$, the lower bound given in (5.1) is better than one in [31, (1.5)].

We recall the definition of "power-type mean". Let $p \in \mathbb{R}$ and M be a bivariate mean. Then the function $M_p : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ defined by

$$(5.2) \quad M_p \equiv M_p(a, b) = M(a^p, b^p)^{1/p} \text{ if } p \neq 0 \text{ and } M_0 = \sqrt{ab}$$

is proved to be a mean (see [32, Theorem 1]), and is called " p -order M mean" or "power-type mean". Also, we have

$$(5.3) \quad M_{p\lambda}(a, b) = M(a^{p\lambda}, b^{p\lambda})^{1/(p\lambda)} = M_p(a^\lambda, b^\lambda)^{1/\lambda}$$

for all $\lambda \in \mathbb{R}$.

Remark 6. The first inequality in (1.9) can be written as $TQ(a, b) < A_{1/2}(a, b)$. Also, it has been proven that $A_{2/3}(a, b)$ is the best lower bound for identric (exponential) mean $\mathcal{I}(a, b)$ (see [33], [34]). Then by taking $M = A$, $p = 2/3$, $\lambda = 3/4$ in identity (5.3) we have

$$TQ(a, b) < A_{1/2}(a, b) = A_{2/3}(a^{3/4}, b^{3/4})^{4/3} < \mathcal{I}(a^{3/4}, b^{3/4})^{4/3} = \mathcal{I}_{3/4}(a, b),$$

which is superior to the second inequality in (1.8), that is,

$$TQ(a, b) < \mathcal{I}_{3/4}(a, b) < \mathcal{I}(a, b),$$

because that the p -order identric (exponential) mean is increasing in $p \in \mathbb{R}$ (see [32]).

Further, expanding in power series gives

$$I_0(t) - \exp\left(\frac{t}{\tanh pt} - 1/p\right) = -\frac{1}{3}\left(p - \frac{3}{4}\right)t^2 + O(t^4),$$

which implies that the condition $p \geq 3/4$ is necessary for the inequality $I_0(t) < \exp(t \coth pt - 1/p)$ to hold for all $t > 0$. This statement can be stated as a theorem.

Theorem 8. For $a, b > 0$ with $a \neq b$, the inequality

$$TQ(a, b) < \mathcal{I}_p(a, b)$$

holds if and only if $p \geq 3/4$.

Remark 7. For the Toader mean $\mathcal{T}(a, b)$ of positive numbers a and b defined by (1.2), it was proved in [8, 12], [35, 36] that

$$(5.4) \quad A_{3/2}(a, b) < \mathcal{T}(a, b) < A_{\ln 2 / \ln(\pi/2)}(a, b)$$

hold for $a, b > 0$ with $a \neq b$, where $2/3$ and $\ln 2 / \ln(\pi/2)$ are the best possible. Very recently, we have shown that

$$(5.5) \quad \mathcal{T}(a, b) < \mathcal{I}_{9/4}(a, b).$$

Replacing (a, b) by $(a^{1/3}, b^{1/3})$ in the first inequality of (5.4) and (5.5), we get that

$$A_{1/2}(a, b)^{1/3} < \mathcal{T}(a^{1/3}, b^{1/3}) = \mathcal{T}_{1/3}(a, b)^{1/3} < \mathcal{I}_{3/4}(a, b)^{1/3},$$

which can be simplified as

$$(5.6) \quad A_{1/2}(a, b) < \mathcal{T}_{1/3}(a, b) < \mathcal{I}_{3/4}(a, b).$$

This together with the inequality $TQ(a, b) < A_{1/2}(a, b)$ gives a nice chain of inequalities:

$$(5.7) \quad TQ(a, b) < A_{1/2}(a, b) < \mathcal{T}_{1/3}(a, b) < \mathcal{I}_{3/4}(a, b).$$

Remark 8. Theorem 4 shows that the double inequality

$$(5.8) \quad L(a, b)^{3/4} A(a, b)^{1/4} < TQ(a, b) < \frac{3}{4}L(a, b) + \frac{1}{4}A(a, b)$$

holds for all $a, b > 0$ with $a \neq b$, where the exponents $3/4$, $1/4$ and weights $3/4$, $1/4$ are the best. Using the well-known inequalities

$$L(a, b) < \frac{A(a, b) + 2G(a, b)}{3}$$

proved in [37] and $A(a, b) > L(a, b)$, we have

$$L(a, b) < L(a, b)^{3/4} A(a, b)^{1/4} < TQ(a, b) < \frac{3}{4}L(a, b) + \frac{1}{4}A(a, b) < \frac{A(a, b) + G(a, b)}{2}.$$

It is thus clear that our inequalities (5.8) improve Qi et al.'s results (1.8) and (1.9).

Remark 9. For the Gauss compound mean $AGM(a, b)$ of positive numbers a and b defined by (1.1), it has been proven in [3], [4], [11, Theorem 1] that

$$(5.9) \quad L(a, b) < AGM(a, b) < L(a, b)^{3/4} A(a, b)^{1/4} < L_{3/2}(a, b)$$

for $a, b > 0$ with $a \neq b$, where $L_p(a, b) = L(a^p, b^p)^{1/p}$ is the p -order logarithmic mean. This in combination with our inequalities (5.8) and (5.7) yields a more nice chain of inequalities involving Gauss compound mean, Toader mean and Toader-Qi mean:

$$(5.10) \quad \begin{aligned} L(a, b) &< AGM(a, b) < L(a, b)^{3/4} A(a, b)^{1/4} < TQ(a, b) \\ &< \frac{3}{4}L(a, b) + \frac{1}{4}A(a, b) < A_{1/2}(a, b) < \mathcal{T}_{1/3}(a, b) < \mathcal{I}_{3/4}(a, b). \end{aligned}$$

Moreover, inspired by the third inequality in 5.9 and the first inequality in 5.8, we propose a conjecture as follows.

Conjecture 1. For $a, b > 0$ with $a \neq b$, the inequality

$$TQ(a, b) > L_{3/2}(a, b)$$

holds.

Remark 10. From Corollary 1 or the chain of inequalities (4.6) we have

$$\sqrt{\cosh t} < I_0(t) < \frac{\cosh t + 1}{2},$$

which is equivalent to

$$\sqrt{A(a, b) G(a, b)} < TQ(a, b) < \frac{A(a, b) + G(a, b)}{2}.$$

This in conjunction with the inequalities

$$\sqrt{A(a, b) G(a, b)} < \sqrt{L(a, b) \mathcal{I}(a, b)} < \frac{L(a, b) + \mathcal{I}(a, b)}{2} < \frac{A(a, b) + G(a, b)}{2}$$

proved in [38] gives rise to another conjecture.

Conjecture 2. For $a, b > 0$ with $a \neq b$, the inequalities

$$\sqrt{A(a, b) G(a, b)} < TQ(a, b) < \sqrt{L(a, b) \mathcal{I}(a, b)} < \frac{L(a, b) + \mathcal{I}(a, b)}{2} < \frac{A(a, b) + G(a, b)}{2}$$

hold.

Remark 11. Kazarinoff in [39] gave the following Wallis inequalities:

$$(5.11) \quad \frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n \in \mathbb{N}.$$

For more information on the Wallis inequalities, please refer to [25] and the references therein. Our Lemma 5 tells us that the sequence $\{s_n\}$ is strictly decreasing for $n \in \mathbb{N}$, therefore, we have

$$\frac{2}{\pi} < s_n = (2n+1) \left(\frac{(2n-1)!!}{2^n n!} \right)^2 < \frac{3}{4},$$

which is equivalent to

$$\frac{1}{\sqrt{\pi} \sqrt{n+1/2}} < \frac{(2n-1)!!}{2^n n!} < \frac{\sqrt{6}}{4\sqrt{n+1/2}}.$$

From the proof of Theorem 3 it is seen that the sequence $\{c_n/d_n\}$ defined by

$$\frac{c_n}{d_n} = \frac{2^{2n} s_n - 1}{2^{2n} - 1}$$

is strictly increasing for $n = 1, 2$ and decreasing for $n \geq 2$. Therefore, we have

$$\min \left(\frac{2}{3}, \frac{2}{\pi} \right) = \min \left(\frac{c_1}{d_1}, \lim_{n \rightarrow \infty} \frac{c_n}{d_n} \right) < \frac{2^{2n} s_n - 1}{2^{2n} - 1} \leq \frac{c_2}{d_2} = \frac{41}{60},$$

which is equivalent to

$$(5.12) \quad \sqrt{\frac{(\pi-2)2^{-2n}+2}{\pi(2n+1)}} < \frac{(2n-1)!!}{(2n)!} < \sqrt{\frac{41+19 \times 2^{-2n}}{60(2n+1)}}, \quad n \in \mathbb{N}.$$

This in fact gives a new Wallis type inequality, and the lower bound given in (5.12) is clearly superior to the one given in (5.11).

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